Docking Dynamics for Rigid-Body Spacecraft

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A generic analysis of rigid-body docking dynamics from initial physical contact of the two spacecraft to final latching is presented The results can be used to estimate the reaction forces and torques on the main craft, and its resulting motion with respect to space for arbitrary contact conditions and a prescribed relative motion of the rendezvous craft The docking interval is separated into three phases: initial contact, continuous maneuver, and latching The first and third phases are analyzed as impulsive, whereas the second phase occurs over a finite time Two simple examples of possible docking maneuvers are presented to illustrate the results

Introduction

VARIOUS proposed space systems require two spacecraft to rendezvous and ultimately dock, i e , to be rigidly tied together in space in order to transfer men, supplies, or equipment Hereafter, the spacecraft which is already in orbit (or conceivably in deeper space) will be designated the main vehicle M, while the second will be called the rendezvous vehicle m An exact analysis of the dynamics of any particular system depends, of course, upon the selected mechanisms for the docking and latching (see, for example, Ward and Williams[1]) The purpose here, however, is not to propose any specific mechanism but rather to present a general analysis which reveals the effect of the significant docking parameters Some of these parameters are attitude misalignments between the vehicles at the initiation of the docking, initial vehicle rates, and the docking rate, i e , the rate at which the vehicles are drawn together Such information is necessary, for example, in designing an attitude control system for main vehicle stabilization during the docking

We consider the docking interval to begin at the time of initial physical contact between the two vehicles, and to end when all relative motion between them has ceased At this time they become a single rigid body latched together in a predetermined fashion Since we are not analyzing any specific technique, we have attempted to present a generalized analysis applicable to many docking techniques Accordingly, it appears that the docking interval may be separated into three more or less distinct phases: the initial contact, a continuous maneuver, and the latching

The initial phase starts with the first physical contact between the two vehicles; in general there will be errors in their linear and angular positions relative to the position they must finally achieve in order to successfully latch together Depending on the precise mechanical arrangement there could very well be a succession of "bumps" and separations However, shortly after contact at least one point on each of the vehicles will be common to both, and a definite docking maneuver will be under way This first phase should be of a relatively short duration and is assumed impulsive Thus, the viewpoint is that at $t = 0$ the vehicles collide with given initial linear and angular velocities and a given relative attitude There is, effectively, an impulsive collision between them involving negligibly small position and attitude changes such that, at the conclusion of the collision, the rendezvous vehicle has a prescribed linear and angular velocity relative to the main vehicle The first problem is to determine the final dynamical state of both vehicles, given the initial state and the prescribed relative rates

The existence of the second phase, the continuous maneuver, is implied by initial position or attitude errors That is, although velocities can be changed more or less impulsively, position or angular changes require a finite time in which to be effected Thus, during the second phase the rendezvous vehicle is moving relative to the main vehicle in translation and rotation From our viewpoint it is necessary to prescribe realistic relative motions in order to pursue the analysis, since no specific system is under consideration The relative motion of the rendezvous vehicle implies the existence of reaction forces and torques on the main vehicle General equations for the dynamics of a main vehicle which contains an arbitrary number of moving parts have been presented, with varying approaches, by Roberson,[2] Abzug,[3] and Grubin[4] Similar equations are presented here for the relatively simple case of only one moving part The second-phase analysis consists of formulating and solving the equations of motion for the main vehicle for a prescribed relative motion of the rendezvous vehicle, using as initial conditions the dynamical state at the conclusion of the first phase The second phase terminates when the two vehicles are in the correct position and attitude for latching

The third phase consists of latching the vehicles together Two dynamical possibilities exist here, depending on the motion during the second phase The first possibility is that a control system has been in operation such that at the time the rendezvous vehicle arrives at the latching position, all relative linear and angular velocities between the two vehicles are zero The latching then becomes trivial dynamically since, in principle, no forces or torques are required to secure the rendezvous vehicle † The other possibility is that relative velocities do exist, so that latching implies reaction forces and torques on the main vehicle as these relative velocities are reduced to zero The latching interval should be short and will be treated here as impulsive The third-phase analysis determines the latching force and torque, and the final motion of the two vehicles together as a single rigid body

The first part of the paper presents the analyses for the three phases; in the second part the results are illustrated by two examples of possible docking schemes The first example illustrates the effect of a position deviation between the initial contact point and the latching point The second example illustrates the effect of an attitude deviation at contact relative to the latching attitude

Presented as Preprint 62-43 at the IAS 30th Annual Meeting, New York, January 22–24, 1962 This paper was prepared while the author was with the Nortronics Division of the Northrop Corporation

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† As a generalization of this, it is theoretically possible for the vehicles to collide with essentially zero relative velocity, and to be in the correct relative position and attitude for the latching In such a case the vehicles can be immediately latched and there are no dynamics at all

Docking Dynamics

Initial Contact

The initial phase is depicted in Fig 1 Prior to contact ($t = 0-$) the two vehicles have arbitrary linear and angular velocities and attitude orientation The points of contact on each vehicle are designated as C_M and C_m, respectively, and of course are in spatial coincidence at $t = 0$ ‡ We assume that the net effect of the initial action of the docking mechanism is to generate impulsive forces and torques between the two vehicles, such that immediately after contact, point C_m has a prescribed velocity $\dot{\mathbf{q}}$ relative to point C_M (although the points are still spatially coincident), and the rendezvous vehicle takes on a prescribed angular velocity $\boldsymbol{\mu}_0$ relative to the main vehicle §

Since there are no external impulsive forces during the first phase, the linear momentum of the system is conserved Similarly, there are no external impulsive torques, so the angular momentum of the system is conserved with respect to any inertial origin or the CMC (combined mass center) We choose as origin for the angular momentum equation an inertial point instantaneously coincident with the contact point Two more conditions follow from the kinematics of the prescribed relative motion To express these conditions analytically we use the notation

M,m = the masses of the main and rendezvous vehicles

$\begin{Bmatrix}\mathbf{U}_0,\mathbf{u}_0\\ \mathbf{V}_0,\mathbf{v}_0\end{Bmatrix}$ = the linear velocities of the mass centers of M and m $\begin{Bmatrix}\text{before}\\ \text{after}\end{Bmatrix}$ the collision‖

$\begin{Bmatrix}\boldsymbol{\Lambda}_0,\boldsymbol{\lambda}_0\\ \boldsymbol{\Omega}_0,\boldsymbol{\omega}_0\end{Bmatrix}$ = the angular velocities of M and m $\begin{Bmatrix}\text{before}\\ \text{after}\end{Bmatrix}$ the collision

$\begin{Bmatrix}\boldsymbol{\Sigma}_0,\boldsymbol{\sigma}_0\\ \mathbf{H}_0,\mathbf{h}_0\end{Bmatrix}$ = the angular momenta for M and m referred to their individual mass centers $\begin{Bmatrix}\text{before}\\ \text{after}\end{Bmatrix}$ the collision

$\mathbf{P}$,$\mathbf{p}$ = the position vectors from the contact point to the individual mass centers

The equation of linear momentum yields

$$M\mathbf{U}_0 + m\mathbf{u}_0 = M\mathbf{V}_0 + m\mathbf{v}_0 \quad (1)$$

The equation of angular momentum yields

$$(\boldsymbol{\Sigma}_0 + M\mathbf{P} \times \mathbf{U}_0) + (\boldsymbol{\sigma}_0 + m\mathbf{p} \times \mathbf{u}_0) = (\mathbf{H}_0 + M\mathbf{P} \times \mathbf{V}_0) + (\mathbf{h}_0 + m\mathbf{p} \times \mathbf{v}_0) \quad (2)$$

The kinematical conditions are

$$[\mathbf{v}_0 + \mathbf{p} \times \boldsymbol{\omega}_0] - [\mathbf{V}_0 + \mathbf{P} \times \boldsymbol{\Omega}_0] = \dot{\mathbf{q}} \quad (3)$$

$$\boldsymbol{\omega}_0 - \boldsymbol{\Omega}_0 = \boldsymbol{\mu}_0 \quad (4)$$

Equations (1–4) are supplemented by the equations relating $\mathbf{H}_0, \mathbf{h}_0$ to $\boldsymbol{\Omega}_0, \boldsymbol{\omega}_0$:

$$\mathbf{H}_0 = \mathbf{E}_j[I_{jk}(\boldsymbol{\Omega}_0 \cdot \mathbf{E}_k)] \quad (5)$$

$$\mathbf{h}_0 = \mathbf{e}_j[i_{jk}(\boldsymbol{\omega}_0 \cdot \mathbf{e}_k)] \qquad (j,k = 1, 2, 3) \quad (6)$$

In Eqs (5) and (6) note first that we are using the summation convention on repeated subscript indices In (5), $\mathbf{E}_j$ are unit vectors along a set of rectangular axes X_j fixed in M with origin at the mass center of M, and I_{jk} is the inertia tensor for M referred to axes X_j # Unit vectors $\mathbf{e}_j$ and inertia tensor i_{jk} have similar meanings with respect to a set of axes x_j fixed in m, whose origin is at the mass center of m Axes X_j, x_j need not be principal axes nor need they be parallel in the general case The angular momenta prior to contact can be similarly expressed:

$$\boldsymbol{\Sigma}_0 = \mathbf{E}_j[I_{jk}(\boldsymbol{\Lambda}_0 \cdot \mathbf{E}_k)] \quad (7)$$

$$\boldsymbol{\sigma}_0 = \mathbf{e}_j[i_{jk}(\boldsymbol{\lambda}_0 \cdot \mathbf{e}_k)] \quad (8)$$

By substitution of Eqs (5–8) into (2), Eqs (1–4) then determine $\mathbf{V}_0$, $\mathbf{v}_0$, $\boldsymbol{\Omega}_0$, and $\boldsymbol{\omega}_0$ as functions of the initial dynamical state and the prescribed relative velocities $\dot{\mathbf{q}}$ and $\boldsymbol{\mu}_0$ Since (1–4) are linear algebraic equations, a general solution could be obtained However, this does not seem particularly useful; instead, specific solutions will be given later in the examples

It is also of interest to determine the impulsive force and impulsive concentrated torque which exist between M and m in order to enforce the prescribed relative motion ** If we designate the force and torque as $\mathbf{I}$ and $\boldsymbol{\tau}$ acting on M, they are determined by applying the equations of linear and angular momentum to M alone These equations are

$$M\mathbf{U}_0 + \mathbf{I}_c = M\mathbf{V}_0 \quad (9)$$

$$\boldsymbol{\Sigma}_0 + (\mathbf{I} \times \mathbf{P}) + \boldsymbol{\tau} = \mathbf{H}_0 \quad (10)$$

where the reference point in (10) is the mass center of M In (9) and (10), $\mathbf{V}_0$ and $\mathbf{H}_0$ are determined by Eqs (1–8), so that $\mathbf{I}_c$ is determined by (9) and then $\boldsymbol{\tau}$ follows from (10) The force and torque on m are, of course, $-\mathbf{I}$, $-\boldsymbol{\tau}$

Summarizing the calculations, Eqs (1–4) supplemented by (5–8) determine the final dynamical variables, and (9) and (10) determine the action-reaction force and torque between the two vehicles For practical calculations a coordinate system needs to be selected for evaluating these vector equations Which system is most convenient depends on the

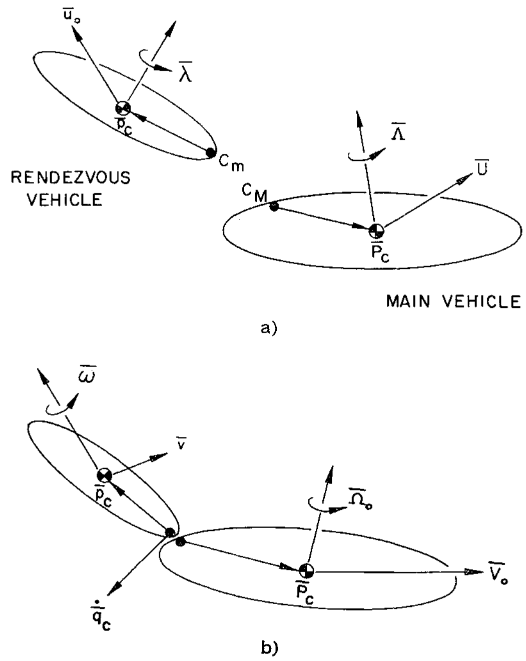


Fig 1 Initial contact: a) before contact, b) after contact

‡ In general, capital letters refer to the main vehicle, smal letters to the rendezvous vehicle

§ Subscript 0 in $\boldsymbol{\mu}_0$ refers to $t = 0$, while subscript c in $\dot{\mathbf{q}}_c$ refers to the contact point Since contact occurs only at $t = 0$, it is unnecessary to write $\dot{\mathbf{q}}_{c0}$

‖ All linear and angular velocities are with respect to inertial space unless otherwise noted

According to the preceding formulation, the products of inertia must be defined as the negatives of the conventional products of inertia; for example, we define $I_{12} = I_{21} = -\int_M X_1X_2dM$

** In the appendix we consider an alternate set of initial dynamical conditions wherein the torque is zero

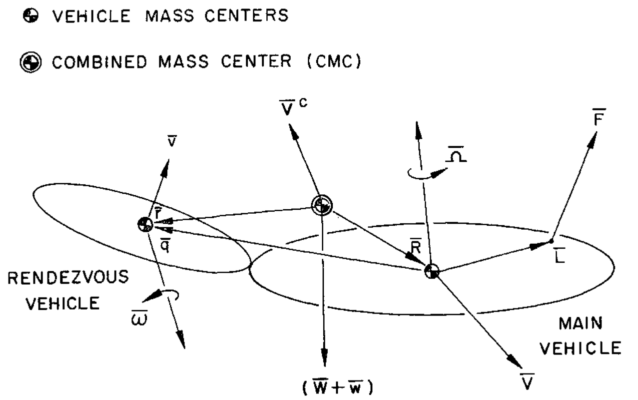


Fig 2 Continuous maneuver

particular problem; a usual choice is the X_j system fixed in the main vehicle

Continuous Maneuver

In the second phase of the docking, the rendezvous vehicle is being continuously maneuvered into the correct linear and angular position relative to the main vehicle in order to arrive at the latching position The initial values of the dynamical variables for the second phase are the final values from the first phase The viewpoint in this section is to formulate the equations of motion for the main vehicle for a prescribed relative motion of the rendezvous vehicle In a complete analysis the dynamical equations for the docking mechanism itself would also be considered, and they would couple in with the equations presented below

Figure 2 shows the schematic upon which the analysis is based The notation is: $\mathbf{V}$, $\mathbf{v}$ are the velocities of the mass centers of M and m, while $\mathbf{\Omega}$, $\boldsymbol{\omega}$ are their angular velocities, where $\mathbf{V}$, $\mathbf{v}$, $\mathbf{\Omega}$, $\boldsymbol{\omega}$ are continuous functions of time Vector $\mathbf{q}$ is the position vector locating the mass center of m relative to M, and $\boldsymbol{\mu}$ describes the angular velocity of m relative to M Thus $\boldsymbol{\omega} = \mathbf{\Omega} + \boldsymbol{\mu}$ It is $\mathbf{q}(t)$ and $\boldsymbol{\mu}(t)$ which describe the docking maneuver ††

Again in Fig 2, vectors $\mathbf{R}(t)$, $\mathbf{r}(t)$ locate the mass centers of M and m relative to their combined mass center (CMC) During the continuous maneuver the CMC wanders with respect to either vehicle owing to the relative motion between them The velocity of the CMC is designated as $\mathbf{V}^C$, and henceforth a superscript C will be used to designate all CMC variables

The other parameters in Fig 2 are the weight forces $\mathbf{W}$ and $\mathbf{w}$; here we assume that the variation in the gravitational field is negligibly small over the volume of the vehicles so that the resultant gravitational force acts at the CMC We include also a control force $\mathbf{F}$ acting on M through lever arm $\mathbf{L}$ as shown If there are other control forces on M (or m), the results given below can be suitably generalized All other natural forces such as, for example, solar radiation pressure are assumed to be negligible We may now formulate the dynamical equations for the system

The equation of linear momentum runs

$$\dot{\mathbf{V}}^C = \mathbf{g} + \mathbf{F}/(M + m) \qquad (11)$$

where $\dot{\mathbf{V}}^C$ is the inertial acceleration of the CMC and $\mathbf{g}$ is the local gravitational acceleration Equation (11) will not yield any further information until $\mathbf{g}$ is specified as a function of position and $\mathbf{F}$ is specified as a function of time or of other variables The initial condition for (11) is

$$(M + m)\mathbf{V}_0{}^C = M\mathbf{V}_0 + m\mathbf{v}_0 = M\mathbf{U}_0 + m\mathbf{u}_0 \qquad (12)$$

upon using Eq (1)

The angular momentum equation written with respect to the CMC is

$$(d/dt)\,[(\mathbf{H} + M\mathbf{R}\times\dot{\mathbf{R}}) + (\mathbf{h} + m\mathbf{r}\times\dot{\mathbf{r}})] = (\mathbf{R} + \mathbf{L})\times\mathbf{F} \qquad (13)$$

where $\mathbf{H}$, $\mathbf{h}$ are as defined in (5) and (6), upon replacing $\mathbf{\Omega}_0$, $\boldsymbol{\omega}_0$ by $\mathbf{\Omega}(t)$, $\boldsymbol{\omega}(t)$

In (13) we can eliminate $\mathbf{R}$ and $\mathbf{r}$ by noting that, from Fig 2,

$$\mathbf{R} + \mathbf{q} = \mathbf{r} \qquad M\mathbf{R} + m\mathbf{r} = 0 \qquad (14)$$

where the second of (14) follows from the definition of the CMC

Solving (14),

$$\mathbf{R} = -(1 + \epsilon)^{-1}\epsilon\mathbf{q} \qquad \mathbf{r} = (1 + \epsilon)^{-1}\mathbf{q} \qquad (15)$$

where $\epsilon = m/M$

Using (15), (13) becomes

$$(d/dt)\,[\mathbf{H} + \mathbf{h} + m(1 + \epsilon)^{-1}\mathbf{q}\times\dot{\mathbf{q}}] = [\mathbf{L} - (1 + \epsilon)^{-1}\epsilon\mathbf{q}]\times\mathbf{F} \qquad (16)$$

where $\dot{\mathbf{q}}$ is given below

Since $\mathbf{H}$ and $\mathbf{h}$ are linear functions of $\mathbf{\Omega}$ and $\boldsymbol{\omega} = \mathbf{\Omega} + \boldsymbol{\mu}$, and $\mathbf{q}(t)$, $\boldsymbol{\mu}(t)$ are prescribed, then (16) becomes the equation for determining $\mathbf{\Omega}(t)$, the angular velocity of the main vehicle When $\mathbf{\Omega}(t)$ is found, the main vehicle attitude can be obtained by some suitable scheme such as Euler angles or direction cosines

If the external forces are always zero during the second phase, (16) yields

$$\mathbf{H} + \mathbf{h} + m(1 + \epsilon)^{-1}\mathbf{q}\times\dot{\mathbf{q}} = \text{const} \qquad (17)$$

The constant in (17) is the value of the system angular momentum with respect to the CMC at $t = 0+$ (or $t = 0-$) In what follows it will be designated as $\mathbf{H}_0{}^C$

When the external forces are not zero, the derivatives in (16) may be explicitly required These are $\dot{\mathbf{H}}$, $\dot{\mathbf{h}}$, and $\ddot{\mathbf{q}}$

To obtain $\dot{\mathbf{H}}$, start with $\mathbf{H}$ as [cf (5)]

$$\mathbf{H} = \mathbf{E}_j(I_{jk}\Omega_k) \qquad (j, k = 1, 2, 3) \qquad (18)$$

where Ω_k is the component of $\mathbf{\Omega}$ on axis X_k, i e , $\Omega_k = \mathbf{\Omega}\ \mathbf{E}_k$

Now $\dot{\mathbf{H}} = (d\mathbf{H}/dt)$ as seen in an inertial system, so that to keep the inertia tensor constant we have, in the usual way,

$$\dot{\mathbf{H}} = \mathbf{\Omega}\times\mathbf{E}_j(I_{jk}\Omega_k) + \mathbf{E}_j(I_{jk}\dot{\Omega}_k) \qquad (19)$$

The expression for $\dot{\mathbf{h}}$ is similar:

$$\dot{\mathbf{h}} = \boldsymbol{\omega}\times\mathbf{e}_j(i_{jk}\omega_k) + \mathbf{e}_j(i_{jk}\dot{\omega}_k) \qquad (20)$$

From the definition of $\mathbf{q}$, it follows that $\dot{\mathbf{q}}$, $\ddot{\mathbf{q}}$ are the velocity and acceleration of the mass center of m relative to that of M, so that

$$\dot{\mathbf{q}} = \mathbf{\Omega}\times\mathbf{q} + \dot{\mathbf{q}}' \qquad (21)$$

$$\ddot{\mathbf{q}} = \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{q}) + \dot{\mathbf{\Omega}}\times\mathbf{q} + 2\mathbf{\Omega}\times\dot{\mathbf{q}}' + \ddot{\mathbf{q}}' \qquad (22)$$

where $\dot{\mathbf{q}}'$, $\ddot{\mathbf{q}}'$ are the velocity and acceleration of m relative to M, as seen in a coordinate system fixed in M

Summarizing, the fundamental equations during the second phase are Eqs (11), and (16) or (17) supplemented by Eqs (18–22) as required These equations are valid until time $t = T$ (T = docking time), when the vehicles are ready for latching

†† More generally, the vector $\mathbf{q}$ can be selected to locate any point in m relative to any other point in M Carrying out the analysis, one finds that the position vector which consistently appears in the final results is the vector between the mass centers Accordingly, $\mathbf{q}$ has been selected as previously

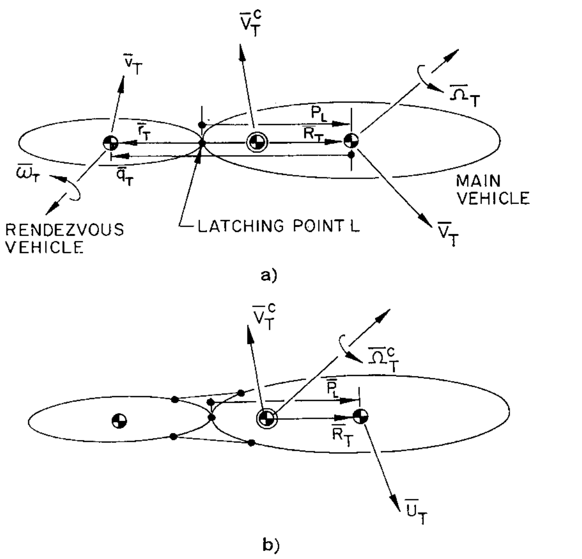


Fig 3 Latching: a) before latching, b) after latching

Latching

The concluding phase of the docking begins when the rendezvous vehicle has arrived at the correct linear position and attitude, relative to the main vehicle, for latching the two vehicles together The over-all problem here is to determine the final dynamical state and the latching force and torque

Figure 3 shows the schematic for the third phase Designating the second-phase interval as $t = 0$ to $t = T$ (we are assuming that if both linear and angular maneuvers were present in the second phase, both are completed in the same time T), we show the linear and angular velocities of M just before latching in Fig 3a as $\mathbf{V}_T \equiv \mathbf{V}(T)$ and $\mathbf{\Omega}_T \equiv \mathbf{\Omega}(T)$, where $\mathbf{\Omega}(T)$ is obtained from the solution to (16) or (17) To obtain $\mathbf{V}_T$ use the definition of $\mathbf{R}$ (Fig 2) and Eq (15):

$$\dot{\mathbf{R}} = \mathbf{V} - \mathbf{V}^C = -(1 + \epsilon)^{-1}\epsilon\dot{\mathbf{q}} \tag{23}$$

so that at $t = T-$

$$\mathbf{V}_T = \mathbf{V}_T{}^C - (1 + \epsilon)^{-1}\epsilon\dot{\mathbf{q}}_T \tag{24}$$

where $\mathbf{V}_T{}^C$ is obtained from the solution to (11) at $t = T$ and $\dot{\mathbf{q}}_T$ follows from (21) as

$$\dot{\mathbf{q}}_T = \mathbf{\Omega}_T \times \mathbf{q}_T + \dot{\mathbf{q}}_T{}' \tag{25}$$

In Eq (25), $\mathbf{q}_T$, $\dot{\mathbf{q}}_T{}'$ are to be evaluated from the prescribed motion $\mathbf{q}(t)$ as $t \to T$ from the left

The linear and angular velocities of m prior to latching are $\mathbf{v}_T$ and $\boldsymbol{\omega}_T = \mathbf{\Omega}_T + \boldsymbol{\mu}_T$ Since $\mathbf{V}_T$, $\mathbf{V}_T{}^C$ are given in the foregoing, $\mathbf{v}_T$ can be obtained from the equation

$$M\mathbf{V}_T + m\mathbf{v}_T = (M + m)\mathbf{V}_T{}^C \tag{26}$$

although it is not needed explicitly in what follows

The other quantities in Fig 3 are $\mathbf{R}_T$, $\mathbf{r}_T$, the position vectors from the CMC to the individual mass centers, and $\mathbf{P}_L$, the position vector from the latching point to the mass center of M

In Fig 3b, the vehicles are shown rigidly latched together The dynamical quantities are $\mathbf{V}_T{}^C$, the linear velocity of the CMC; $\mathbf{\Omega}_T{}^C$, the angular velocity of the combined system; and $\mathbf{U}_T$, the velocity of the mass center of M The angular velocity of M is, of course, $\mathbf{\Omega}_T{}^C$

The latching phase, like the contact phase, is assumed to have negligible duration Since there are no external impulsive forces, the equation of linear momentum immediately yields the result that $\mathbf{V}^C(t)$ is continuous at $t = T$, as already indicated in Figs 3a and 3b

Similarly, since there are no external impulsive torques, the angular momentum with respect to the CMC is also conserved Now from (16) the angular momentum at $t = T-$ is

$$\mathbf{H}_T + \mathbf{h}_T + m(1 + \epsilon)^{-1}\mathbf{q}_T \times \dot{\mathbf{q}}_T \tag{27}$$

whereas at $t = T+$ the vehicles constitute a single rigid body with angular momentum

$$\mathbf{E}_j{}^C[I_{jk}{}^C(\mathbf{\Omega}_T{}^C \cdot \mathbf{E}_k{}^C)] \tag{28}$$

where $\mathbf{E}_j{}^C$ are unit vectors along a set of rectangular axes $X_j{}^C$ fixed in the combined vehicle with origin at the CMC, and $I_{jk}{}^C$ is the inertia tensor for the combined vehicle referred to these axes $I_{jk}{}^C$ is known since the attitude in which the vehicles are latched is known Equating (27) and (28),

$$\mathbf{H}_T + \mathbf{h}_T + m(1 + \epsilon)^{-1}\mathbf{q}_T \times \dot{\mathbf{q}}_T = \mathbf{E}_j{}^C[I_{jk}{}^C(\mathbf{\Omega}_T{}^C \cdot \mathbf{E}_k{}^C)] \tag{29}$$

Since all other terms are known in (29), the final angular velocity $\mathbf{\Omega}_T{}^C$ can be found

To determine the concluding impulsive force and impulsive concentrated torque applied at the latching point, apply the equations of impulsive motion to M only Designating the force as $\mathbf{I}_L$ acting on M, the equation of linear momentum yields

$$M\mathbf{V}_T + \mathbf{I}_L = M\mathbf{U}_T = M(\mathbf{V}_T{}^C + \mathbf{\Omega}_T{}^C \times \mathbf{R}_T) \tag{30}$$

and substituting from (24) for $(\mathbf{V}_T{}^C - \mathbf{V}_T)$,

$$\mathbf{I}_L = m(1 + \epsilon)^{-1}\dot{\mathbf{q}}_T + M(\mathbf{\Omega}_T{}^C \times \mathbf{R}_T) \tag{31}$$

In (31), $\dot{\mathbf{q}}_T$ is given by (25) and $\mathbf{\Omega}_T{}^C$ is obtained from (29) Thus $\mathbf{I}_L$ is explicitly determined

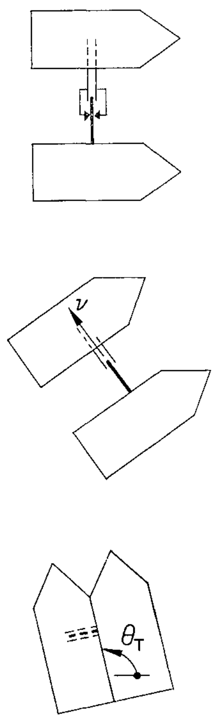
To obtain the torque $\boldsymbol{\tau}_L$, write the angular momentum equation for M using its mass center as origin Thus,

$$\mathbf{H}_T + \mathbf{I}_L \times \mathbf{P}_L + \boldsymbol{\tau}_L = \mathbf{\Sigma}_T \tag{32}$$

where $\mathbf{H}_T$ is defined by (18) with $t = T$ and $\mathbf{\Sigma}_T$ is the angular momentum of M referred to its own mass center at the conclusion of the latching Since the angular velocity of M is $\mathbf{\Omega}_T{}^C$,

$$\mathbf{\Sigma}_T = \mathbf{E}_j[I_{jk}(\mathbf{\Omega}_T{}^C \cdot \mathbf{E}_k)] \tag{33}$$

Also, $\mathbf{P}_L$ is known and $\mathbf{I}_L$ is determined in (31), so that (32) determines $\boldsymbol{\tau}_L$



a) Initial contact

b) Continuous maneuver

c) Latching

Fig 4 Example 1

Summarizing the third-phase calculations, the linear velocity of the CMC is obtained from (11) at $t = T$, the final angular velocity is obtained from (29), and the latching action-reaction force and torque are obtained from (31) and (32)

This completes the basic calculations according to our conception of the docking procedure The vector equations are not too complicated, although component calculations for a three-dimensional example can become laborious In what follows we will illustrate solution of the equations for two specific docking maneuvers

Example 1

We consider for our first example the system shown in Fig 4 Two vehicles of identical shape, mass, and moment of inertia approach each other as shown in Fig 4a At the instant that the lower vehicle is directly underneath, a clamping arrangement housed in the extended boom of the upper vehicle grabs the boom on the lower vehicle so that all relative motion between the two vehicles effectively ceases except in the direction perpendicular to their longitudinal axes The assumed docking motion, Fig 4b, consists of drawing the vehicles together into a final side by side configuration as shown in Fig 4c Insofar as drawing the vehicles together is concerned, this scheme was adopted from one proposed by Kamm,[5] but we emphasize that otherwise what is considered here is not intended to represent Ref 5

Figure 5 details all significant geometry, axes, and assumed initial conditions prior to contact Only a two-dimensional case is being considered; thus, all position and linear velocity vectors have two components, whereas angular velocities and angular momenta have only one component, which is perpendicular to the plane of the paper The coordinate system chosen is a set of axes fixed in the upper vehicle (the main vehicle), the unit vectors along these axes being $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ For brevity, we will designate the components of any two-dimensional vector $\mathbf{A}$ as

$$\mathbf{A} \equiv [A_1, A_2] \tag{34a}$$

and any one-dimensional vector $\mathbf{B}$ as

$$\mathbf{B} \equiv [B_3] \tag{34b}$$

The assumed conditions prior to contact are that both vehicles have identical linear velocities U_0 along $\mathbf{E}_1$; the upper vehicle has zero angular velocity, whereas the lower has angular velocity λ_0, and the vehicles are parallel as regards their attitude The mass of each vehicle is M, and the moment of inertia of each vehicle about its own mass center is I The docking problem is to draw the lower vehicle toward the upper vehicle (or the other way) until the two are in their final side-by side configuration as shown in Fig 4c

Initial Contact

Using the notation for vectors introduced in Eq (34) and comparing Figs 1 and 5, the initial data are

$$\begin{aligned} \mathbf{U}_0 &= [U_0, 0] & \mathbf{u}_0 &= [U_0, 0] \\ \mathbf{\Lambda}_0 &= [0] & \boldsymbol{\lambda}_0 &= [\lambda_0] \\ \mathbf{P} &= [-a, b] & \mathbf{p} &= [-a, -b] \end{aligned} \tag{35a}$$

The initial relative motion is prescribed as

$$\dot{\mathbf{q}} = [0, \nu] \qquad \boldsymbol{\mu}_0 = [0] \tag{35b}$$

The unknowns to be determined are

$$\begin{aligned} \mathbf{V}_0 &\equiv [V_1, V_2] \\ \mathbf{v}_0 &\equiv [v_1, v_2] \\ \mathbf{\Omega}_0 &= \boldsymbol{\omega}_0 = [\Omega_0] \end{aligned} \tag{36}$$

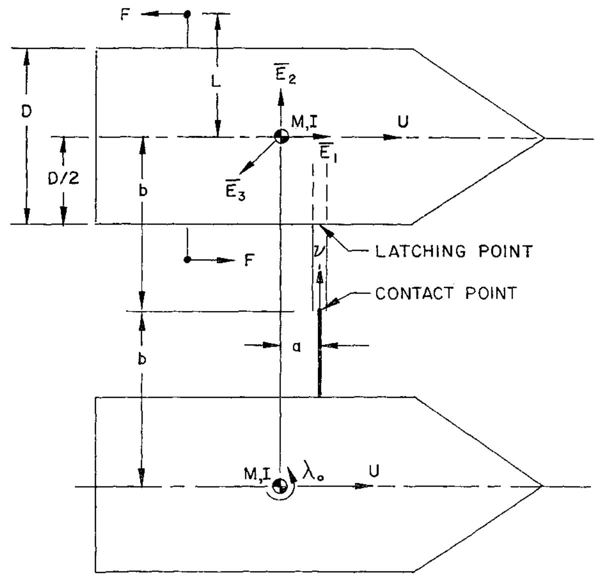


Fig 5 Detailed configuration for example 1

Using (35) and (36) in Eqs (1–8), we obtain

$$\begin{aligned} V_1 &= U_0 - b\Omega_0 & V_2 &= -\nu/2 \\ v_1 &= U_0 + b\Omega_0 & v_2 &= \nu/2 \\ \Omega_0 &= I\lambda_0/2(I + Mb^2) \end{aligned} \tag{37}$$

The results in (37) show that after the initial contact the two vehicles are effectively rotating around the CMC at rate Ω_0, although they are not rigid but are approaching each other at the relative rate ν

Using Eqs (35–37) in (9) and (10), we determine the impulsive reaction force and torque on M as

$$\begin{aligned} \mathbf{I} &= [-Mb\Omega_0, -M\nu/2] \\ \boldsymbol{\tau} &= [(I + Mb^2)\Omega_0 + Ma\nu/2)] \end{aligned}$$

where Ω_0 is given in (37)

Continuous Maneuver

Let us first neglect the control forces F shown in Fig 5 Then (11) yields the simple result $\dot{\mathbf{V}}^C = \mathbf{g}$ Since the linear motion of the CMC is of little interest here, we will not consider this equation further

The equation for determining the angular velocity of the upper vehicle (in fact both vehicles for this case) is furnished by (17) We assume that the lower vehicle is pulled toward the upper vehicle at the constant rate ν, the same value as at $t = 0+$, while the relative angular velocity between the vehicles remains zero Then the functions $\mathbf{q}(t)$ and $\boldsymbol{\mu}(t)$ are

$$\mathbf{q}(t) = [0, -(2b - \nu t)] \qquad \boldsymbol{\mu}(t) = [0] \tag{38}$$

Then from (18, 21, and 38) there follows,

$$\mathbf{H} = \mathbf{h} = [I\Omega] \qquad \dot{\mathbf{q}} = [(2b - \nu t)\Omega, \nu] \tag{39}$$

where $\Omega = \Omega(t)$ is to be determined

Using (39) in (17) with $\epsilon = 1$, one obtains

$$\Omega[2I + (M/2)(2b - \nu t)^2] = H_0{}^C = I\lambda_0 \tag{40}$$

Equation (40) gives the angular velocity of the two vehicles during the time that they are being drawn together The coefficient of Ω is the instantaneous moment of inertia of the two vehicles about the CMC; since it is decreasing as the vehicles approach each other, Ω must increase to preserve the system angular momentum—a well-known effect Introduc-

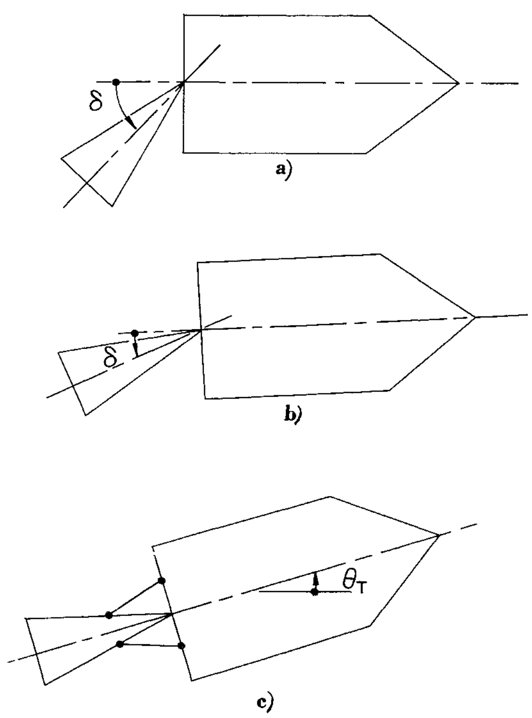


Fig 6 Example 2: a) initial contact, b) continuous maneuver, c) latching

ing θ as the attitude angle $\dot{\theta} = \Omega$, then

$$\theta_T = \int_0^T \Omega dt \tag{41}$$

where θ_T is the total attitude change of the two vehicles as shown in Fig 4c, T is the docking time such that

$$\nu T = 2b - D \tag{42}$$

and D is the transverse dimension of each vehicle as shown in Fig 5

Substituting for Ω from (40) into (41) and integrating, we obtain

$$\theta_T = \left[\frac{H_0{}^C}{Mr_g\nu}\right] \tan^{-1}\left[\frac{r_g(2b - D)}{2r_g{}^2 + bD}\right] \quad (0 < \tan^{-1} < \pi/2) \tag{43}$$

where $r_g = (I/M)^{1/2}$ is the radius of gyration for each vehicle

Equation (43) shows first the result that θ_T is directly proportional to the initial system angular momentum $H_0{}^C$ This implies that rather tight terminal guidance and control requirements may be necessary to minimize relative vehicle rates (and hence system angular momentum) at the initiation of docking More likely, control torques would be required in this case for main vehicle stabilization during docking

Equation (43) also shows that the total attitude change θ_T is inversely proportional to the docking rate ν From (42), for given dimensions b,D, the docking time is also inversely proportional to ν, so θ_T is directly proportional to the docking time This result is reasonable, since from Eq (40), $\Omega(t)$ increases monotonically from Ω_0 given by (37) to $\Omega_T = H_0{}^C(2I + MD^2/2)^{-1}$ independent of ν Hence, if the docking interval is stretched out to a longer time, a larger attitude change must result

Finally, it is interesting to examine the possibility of introducing control torques to reduce the attitude change Consider proportional control forces F always parallel to the main vehicle longitudinal axis (Fig 5) such that

$$F = -(k_\theta \theta + k_\Omega \Omega) \tag{44}$$

acting through lever arms L

Including two forces and noting (40), we find that Eq (16) then yields

$$(d/dt)(I^C\Omega) = -2L(k_\theta \theta + k_\Omega \Omega) \tag{45}$$

where $I^C = I^C(t) = 2I + (M/2)(2b - \nu t)^2$

Carrying out the differentiations in (45), noting $\dot{\theta} = \Omega$, and rearranging, we obtain

$$I^C\ddot{\theta} + [K_\Omega - M\nu(2b - \nu t)]\dot{\theta} + K_\theta \theta = 0 \tag{46}$$

where $K_\theta = 2Lk_\theta$, $K_\Omega = 2Lk_\Omega$, and the initial conditions are $\theta_0 = 0$, $\dot{\theta}_0 = \Omega_0$ which is given by (37)

For a given configuration, solution of (46) determines $\theta(t)$ as a function of the attitude and rate gains K_θ, K_Ω for a given docking rate Note that, according to (45) I^C, the coefficient of $\ddot{\theta}$ is a function of time as is the coefficient of $\dot{\theta}$ Further, since the control forces F would be rocket forces, then the values of M and I are decreasing as mass is expelled However, this decrease will be negligible and constant values for M and I can be assumed

Latching

We will consider the latching dynamics only for the case where the control force F is zero Thus the vehicles are approaching each other "bioadside" at a relative speed ν and at the same time are rotating with an increasing rate Ω given by (40) At time T the latching mechanism must null the relative velocity It is intuitively clear that the forces which accomplish this will not affect the existing angular velocity Ω_T, but this is simple enough to verify using Eq (40) at $t = T$, which equation is equivalent to (29) in this case Thus the angular rate of the system after latching $\Omega_T{}^C$ is

$$\Omega_T{}^C = H_0{}^C/I_T{}^C = H_0{}^C/(2I + MD^2/2) = \Omega_T \tag{47}$$

Hence, until further torques are applied, the two vehicles will rotate together at $\Omega_T{}^C$

The latching forces and torques can be obtained from (31) and (32) We obtain first the vectors

$$\begin{aligned} \dot{\mathbf{q}}_T &= [\Omega_T D, \nu] & \mathbf{R}_T &= [0, D/2] \\ \boldsymbol{\Omega}_T{}^C &= [\text{Eq } (47)] & \mathbf{P}_L &= [-a, D/2] \end{aligned} \tag{48}$$

Substituting (48) in (31–33) yields

$$\mathbf{I}_L = [0, M\nu/2]$$

$$\boldsymbol{\tau}_L = [-M\nu a/2]$$

Note that $\boldsymbol{\tau}_L$ is zero if distance a is zero This simple conclusion, as well as others which can be drawn from the preceding results, would be of direct interest to a mechanical engineer consideiing the design of the proposed docking system or one similar to it

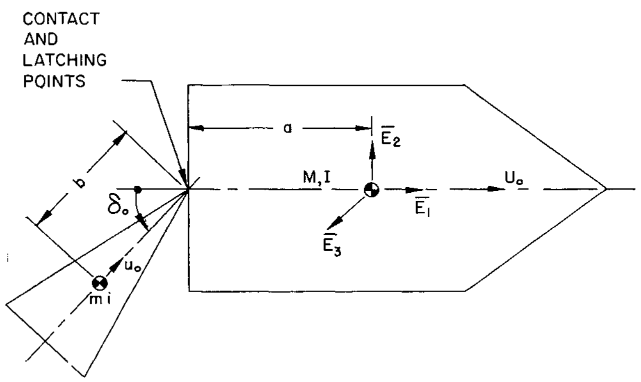


Fig 7 Detailed configuration for example 2

Example 2

For our second example we consider the system of Fig 6 Here we have a front-to-back docking arrangement The smaller vehicle initially butts against the larger one as shown in Fig 6a The docking mechanism here is required to rotate the smaller vehicle as in Fig 6b into final alignment with the larger one prior to the latching shown in Fig 6c In the general case there can be a displacement error between the contact point and the latching point, in addition to the attitude error δ_0 shown in Fig 6a However, we will consider the case where only δ_0 is nonzero while the contact point is coincident with the latching point Only the two-dimensional problem will be examined

Figure 7 details the geometry and initial conditions It is assumed that at the time of contact both vehicles are nonrotating and have velocities U_0, u_0 along their respective longitudinal axes, and that vehicle m is misaligned by angle δ_0 The docking mechanism here is supposed to effectively seize the tip of m and pin it rigidly to M, while at the same time a mechanical torque is exerted on m to initiate the rotation to the latching attitude When this is achieved, the latching mechanism will null any relative rates at this time

Initial Contact

We use the notation of Eqs (34), where the coordinate system is indicated by the unit vectors $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ in Fig 7 Comparing Figs 1 and 7, the initial data are

$$\mathbf{U}_0 = [U_0, 0] \qquad \mathbf{P} = [a, 0]$$

$$\mathbf{u}_0 = [u_0 \cos\delta_0, u_0 \sin\delta_0] \qquad \mathbf{p} = [-b\cos\delta_0, -b\sin\delta_0] \tag{49a}$$

$$\mathbf{\Lambda}_0 = \boldsymbol{\lambda}_0 = [0]$$

The initial relative motion is prescribed as

$$\dot{\mathbf{q}}_c = [0,0] \qquad \boldsymbol{\mu}_0 = [-\mu_0] \tag{49b}$$

The masses and inertias of the two vehicles are M, I, and m, i, respectively, as shown in Fig 7 The unknowns to be determined are

$$\begin{aligned} \mathbf{V}_0 &\equiv [V_1, V_2] \qquad & \mathbf{v}_0 &= [v_1, v_2] \\ \mathbf{\Omega}_0 &= [\Omega_0] \qquad & \boldsymbol{\omega}_0 &= [\Omega_0 - \mu_0] \end{aligned} \tag{50}$$

Using (49) and (50) in Eqs (1–8), one obtains

$$\begin{aligned} V_1 &= (1+\epsilon)^{-1}[U_0 + \epsilon u_0\cos\delta_0 - \epsilon b(\Omega_0 - \mu_0)\sin\delta_0] \\ V_2 &= \epsilon(1+\epsilon)^{-1}[u_0\sin\delta_0 + a\Omega_0 + b(\Omega_0 - \mu_0)\cos\delta_0] \\ v_1 &= (1+\epsilon)^{-1}[U_0 + \epsilon u_0\cos\delta_0 + b(\Omega_0 - \mu_0)\sin\delta_0] \\ v_2 &= (1+\epsilon)^{-1}[\epsilon u_0\sin\delta_0 - a\Omega_0 - b(\Omega_0 - \mu_0)\cos\delta_0] \end{aligned} \tag{51a}$$

where $\epsilon = m/M$ and Ω_0 is determined from the equation

$$I_0^C\Omega_0 = i_0\mu_0 + H_0^C \tag{51b}$$

where

$I_0^C = I + i + m(1+\epsilon)^{-1}(a^2 + b^2 + 2ab\cos\delta_0)$, the moment of inertia of both vehicles about the CMC at $t = 0$

$i_0 = i + m(1+\epsilon)^{-1}b(a\cos\delta_0 + b)$

$H_0^C = -m(1+\epsilon)^{-1}(au_0 + bU_0)\sin\delta_0$, the angular momentum of the system with respect to the CMC at $t = 0$ for the assumed initial conditions [For other initial conditions the mathematical expression for H_0^C will be different but its physical significance in (51b) remains the same]

From (9) and (10) the impulsive force and torque at the contact point are

$$\mathbf{I}_c = [M(V_1 - U_0), MV_2] \qquad \boldsymbol{\tau}_c = [I\Omega_0 + MV_2 a]$$

where V_1, V_2, and Ω_0 are given by Eqs (51)

Continuous Maneuver

Here we consider the external forces always to be zero Thus Eq (11) yields the simple result that $\dot{\mathbf{V}}^C = \mathbf{g}$ during the docking As with the first example, we will not pursue this equation further The rotational equation for the main vehicle is furnished by (17) For the continuous relative motion of m we will consider the case where $\dot{\delta} = \text{const} = -\mu_0$, i e , m is rotated into alignment with M at a constant rate equal to the rate prescribed at the conclusion of the initial contact Thus the relative motion of m during the second phase is described by

$$\mathbf{q} = [-(a + b\cos\delta), -b\sin\delta] \tag{52a}$$

$$\boldsymbol{\mu}(t) = \text{const} = [-\mu_0]$$

where

$$\delta = \delta(t) = \delta_0 - \mu_0 t \tag{52b}$$

and where δ is positive as in Fig 6b

Using (18), we form $\mathbf{H}$ and $\mathbf{h}$, and using (21) we form $\dot{\mathbf{q}}$ Note that

$$\dot{\mathbf{q}}' = b\mu_0[-\sin\delta, \cos\delta] \tag{52c}$$

Substituting the foregoing in (17) yields

$$I^C(t)\Omega = i(t)\mu_0 + H_0^C \tag{53}$$

where

$I^C(t) = I + i + m(1+\epsilon)^{-1}[a^2 + b^2 + 2ab\cos\delta(t)]$, the instantaneous moment of inertia of the system about the instantaneous CMC

$i(t) = i + m(1+\epsilon)^{-1}b[a\cos\delta(t) + b]$

$H_0^C =$ the same as in (51b) [Note the similarity between (53) and (51b) In fact, (53) at $t = 0$ becomes identical to (51b), a result which is clear physically]

Equation (53) determines $\Omega(t)$ during the second phase, $0 < t < T$, where T is the docking time such that

$$\delta_0 - \mu_0 T = 0 \tag{54}$$

Now introduce attitude angle θ such that $\dot{\theta} = \Omega(t)$ Then the total attitude change of the main vehicle during the second phase is

$$\theta_T = \int_0^T \Omega dt \tag{55}$$

Substituting for Ω from (53) and carrying out the integration, we obtain, after simplification,

$$\theta_T = (\delta_0/2) + (1/I_1)[(2H_0^C/\mu_0) + I_2] \times \tan^{-1}[(I_1/I_T^C)\tan(\delta_0/2)] \qquad (0 < \tan^{-1} < \pi/2) \tag{56}$$

where

$I_1 = [(I+i)^2 + 2(I+i)m(1+\epsilon)^{-1}(a^2+b^2) + m^2(1+\epsilon)^{-2}(a^2-b^2)^2]^{1/2}$

$I_2 = i - I + m(1+\epsilon)^{-1}(b^2 - a^2)$

$I_T^C = I^C(t)]_{t=T}$

where $I^C(t)$ is given under (53)

Inspection of (56) reveals first that if H_0^C were zero and also if $i = I$, $b = a$ so that $I_2 = 0$, then $\theta_T = (\delta_0/2)$; i e , with respect to inertial space each vehicle would rotate half the initial attitude error Further, if H_0^C is zero, θ_T is independent of the docking rate μ_0, and then θ_T will be greater than or less than $(\delta_0/2)$ according as $I_2 > 0$ or $I_2 < 0$

From the definitions of I_1, I_2, and I_T^C it is not difficult to establish that $|I_2|/I_1 < 1$ and $I_1/I_T^C < 1$ From the latter condition it follows that the $\tan^{-1}$ in Eq (56) is $<(\delta_0/2)$

Thus again, if $H_0^C = 0$, it follows that even if $I_2 < 0$, $\theta_T > 0$, the case depicted in Fig 6 This is consistent with (53), which shows $\Omega(t)$ always > 0 during the docking if $H_0^C = 0$ ‡‡

Finally, if H_0^C is not zero, then its contribution to θ_T is inversely proportional to the docking rate μ_0 A similar result was noted in example 1 With θ_T determined, the attitude change for m is $(\theta_T - \delta_0)$, positive in the counterclockwise direction

Latching

For the assumed docking maneuver, vehicle m will still have the constant relative rate μ_0 when the vehicles are ready for latching; this rate must be nulled by the latching forces and torques

The final angular rate of the rigid system can be determined by formal application of Eq (29), but since external torques have always been zero it follows immediately that

$$\Omega_T{}^C = H_0{}^C/I_T{}^C \tag{57}$$

where $I_T{}^C$ has been noted under (56)

The latching impulsive force and torque are given by Eqs (31) and (32) The vectors needed here are

$$\begin{aligned} \dot{\mathbf{q}}_T &= [0, b\mu_0 - (a+b)\Omega_T] \\ \mathbf{\Omega}_T{}^C &= [\text{as in (57)}] \\ \mathbf{R}_T &= [\epsilon(1+\epsilon)^{-1}(a+b), 0] \\ \mathbf{P}_L &= [a, 0] \end{aligned} \tag{58}$$

where Ω_T is obtained from (53) applied at $t = T$, namely,

$$I_T{}^C\Omega_T = i_T\mu_0 + H_0{}^C \tag{59}$$

where $i_T = i(t)]_{t=T}$, where $i(t)$ is given under (53)

Combining Eqs (58) and (59) into (31) and (32) and simplifying, one obtains

$$\begin{aligned} \mathbf{I}_L &= [0, m(1+\epsilon)^{-1}(\mu_0/I_T{}^C)(bI - ai)] \\ \boldsymbol{\tau}_L &= [-(\mu_0/I_T{}^C)\{Ii + m(1+\epsilon)^{-1}(b^2I + a^2i)\}] \end{aligned} \tag{60}$$

Equation (60) shows the interesting result that the latching force is zero if $(b/a) = (i/I)$ However, the latching torque is zero only if the closing relative angular rate is zero, a result which is entirely reasonable

‡‡ For the specific initial conditions of (49a), $H_0^C < 0$ as given in (51b) Thus whether $\theta_T \gtrless 0$ for this case depends strongly on the magnitude of μ_0

Appendix

We consider briefly a modified set of dynamical conditions for the contact phase Specificially, suppose the main vehicle mechanism which torques the rendezvous vehicle acts relatively slowly so that the assumption of an initial torque, as required by the prescribed $\boldsymbol{\mu}_0$, is no longer justified Instead, let us replace the prescription of $\boldsymbol{\mu}_0$ by the condition that the initial impulsive concentrated torque is zero Of the original system of Eqs (1–4), (1) and (3) remain while (2) and (4) are replaced by the condition of angular momentum conservation for *each* vehicle with respect to an inertial point coincident with the contact point This follows since the contact torque is now zero, and the contact forces have a zero lever arm with respect to the contact point It was precisely the existence of the contact torque in the original Eqs (1–4) that implied the conservation of angular momentum only for the entire system and not for each vehicle separately Equations (1), (3) and the new Eqs (2) and (4) will again determine the final dynamical variables $\mathbf{V}_0$, $\mathbf{v}_0$, $\mathbf{\Omega}_0$, $\boldsymbol{\omega}_0$ and hence the initial relative angular velocity $(\boldsymbol{\omega}_0 - \mathbf{\Omega}_0)$ Then in prescribing the angular velocity $\boldsymbol{\mu}(t)$ during the second phase, kinematical consistency requires that

$$\lim_{t\to 0+} \boldsymbol{\mu}(t) = \boldsymbol{\omega}_0 - \mathbf{\Omega}_0$$

as just determined

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